Hadamard powers of polynomials with only real zeros

Yi Wang*, Bin Zhang

School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

Abstract

Let \( f(x) = \sum_{i=0}^{n} a_i x^i \) be a polynomial with positive coefficients and \( p > 0 \). The \( p \)-th Hadamard power of \( f(x) \) is the polynomial \( f[p](x) = \sum_{i=0}^{n} a_i^p x^i \). It is conjectured that if \( f(x) \) has only real zeros, then so does \( f[p](x) \) for \( p \geq 1 \). We verify the conjecture when \( n = 3 \) and give a counterexample when \( n = 4 \). We also show that there exists a positive number \( P_n \) such that if \( f(x) \) has only real zeros, then so does \( f[p](x) \) for \( p > P_n \).

AMS classification: 26C10, 15B48

Keywords: Polynomial with only real zeros, Hadamard power, Totally nonnegative matrix, Pólya frequency sequence

1. Introduction

Let \( f(x) = \sum_{i=0}^{n} a_i x^i \) and \( g(x) = \sum_{i=0}^{n} b_i x^i \) be two polynomials of equal degree. Their Hadamard product is the polynomial \( (f \circ g)(x) = \sum_{i=0}^{n} a_i b_i x^i \). A classical result of Maló states that if all zeros of both \( f(x) \) and \( g(x) \) are real and those of \( g(x) \) have the same sign, then all zeros of \( (f \circ g)(x) \) are real as well (see, e.g., [8, Problem 155, Part V]). Denote by \( P_n \) the set of polynomials of degree \( n \) with positive coefficients. Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in P_n \) and \( p > 0 \). Following [4], the \( p \)-th Hadamard power of \( f(x) \) is defined as the polynomial \( f[p](x) = \sum_{i=0}^{n} a_i^p x^i \). We say that \( f(x) \) is a Hurwitz polynomial if all its zeros have negative real parts. It is known that the \( p \)-th Hadamard power of a Hurwitz polynomial is still a Hurwitz polynomial for all real \( p \geq 1 \) [4, Theorem 5]. The similar statement is conjectured to hold for polynomials all of whose zeros are negative real numbers (see, e.g., Fisk [3, Question 10, p. 722]). Gregor and Tišer [4, Theorem 3] gave a questionable argument. Indeed, if a polynomial \( f(x) \) has only real zeros of like sign, then so does the \( p \)-th Hadamard power \( f[p](x) \) for each integer \( p > 1 \) by Maló’s result. However, such a result can not extend to real \( p \). For example, let

\[
 f(x) = (x + 10)(x + 11)(x + 12)(x + 13) = 17160 + 6026x + 791x^2 + 46x^3 + x^4
\]

This work was supported in part by the National Natural Science Foundation of China (Nos. 10771027, 11071030) and the Specialized Research Fund for the Doctoral Program of Higher Education of China (No. 20110041110039).

*Corresponding author

Email addresses: wangyi@dlut.edu.cn (Yi Wang), redzhang1115@163.com (Bin Zhang)

Preprint submitted to LAA August 23, 2013
and \( p = 1.139 \). Then \( f^{[p]}(x) \) has nonreal zeros. Actually, four zeros of \( f^{[p]}(x) \) are approximate to

\[-39.7018, -6.49632, -16.0617 \pm 0.178468i\]

using Mathematica 8.0.

Throughout this paper all polynomials considered have positive coefficients. We show that if a cubic polynomial \( f(x) \) has only real zeros, then so does \( f^{[p]}(x) \) for real \( p > 1 \). We also show that there exists a real number \( P_n > 1 \) such that if a polynomial \( f(x) \) of degree \( n \) has only real zeros, then so does \( f^{[p]}(x) \) for \( p > P_n \).

2. Main results

We present our main results in this section.

**Theorem 1.** Let \( f(x) \) be a cubic polynomial with positive coefficients. If \( f(x) \) has only real zeros, then so does its \( p \)-th Hadamard power \( f^{[p]}(x) \) for real \( p > 1 \).

**Proof.** We distinguish three cases to show that three zeros of \( f^{[p]}(x) \) are all real.

**Case 1.** Suppose that \( f(x) \) has a zero of multiplicity 3. Then we may assume, without loss of generality, that \( f(x) = (x + 1)^3 \). Thus

\[ f^{[p]}(x) = x^3 + 3p^2x^2 + 3px + 1 = (x + 1)[x^2 + (3p - 1)x + 1]. \]

The quadratic factor \( x^2 + (3p - 1)x + 1 \) has two real zeros for \( p > 1 \) since its discriminant \((3p - 1)^2 - 4 > 0 \). Three zeros of \( f^{[p]}(x) \) are therefore all real.

**Case 2.** Suppose that \( f(x) \) has a zero of multiplicity 2. Then we may assume that \( f(x) = (x + 1)^2(x + r) \), where \( r > 0 \) and \( r \neq 1 \). Let \( f^*(x) = x^3f(1/x) \) be the reciprocal polynomial of the polynomial \( f(x) \). Then \( f^*(x) = r(x + 1)^2(x + 1/r) \). Note that \((f^{[p]})^*(x) = (f^*)^{[p]}(x) \) and that a polynomial has only real zeros if and only if its reciprocal polynomial has the same property. So it suffices to consider the case \( r > 1 \).

Now \( f^{[p]}(x) = x^3 + (r + 2)^p x^2 + (2r + 1)^p x + r^p \). Let

\[ x_0 := \frac{(2r + 1)^p}{(r + 2)^p}. \]

Then \( x_0 < -1 \). We next show that \( f^{[p]}(x_0) > 0 \) and \( f^{[p]}(-1) < 0 \), which imply that \( f^{[p]}(x) \) has one real zero in each of three intervals \((−\infty, x_0), (x_0, -1) \) and \((-1, 0)\).

Actually, we have

\[ f^{[p]}(x_0) = f^{[p]} \left( \frac{(2r + 1)^p}{(r + 2)^p} \right) = -\frac{(2r + 1)^{3p}}{(r + 2)^{3p}} + r^p \frac{r^p(r + 2)^{3p} - (2r + 1)^{3p}}{(r + 2)^{3p}} > 0 \]

since \( r(r + 2)^3 - (2r + 1)^3 = (r - 1)^3(r + 1) > 0 \), i.e., \( r(r + 2)^3 > (2r + 1)^3 \). On the other hand, let \( g(x) := x^3 + (2r + 2 - x)^p \). Then \( g'(x) = p[x^{p-1} - (2r + 2 - x)^{p-1}] < 0 \) for \( 0 < x < r + 1 \) and \( p > 1 \). Thus \( g(x) \) is monotonically decreasing in the interval \((0, r + 1)\), and so \( f^{[p]}(-1) = [(r + 2)^p + r^p] - [(2r + 1)^p + 1] = g(r) - g(1) < 0 \), as desired.

**Case 3.** Suppose that \( f(x) \) has three distinct real zeros. Let \( f(x) = ax^3 + bx^2 + cx + d \). Then its discriminant is \( \Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2 \). It is well-known that
(i) if \( \Delta > 0 \), then \( f(x) \) has three distinct real zeros;
(ii) if \( \Delta = 0 \), then \( f(x) \) has a multiple zero and all its zeros are real;
(iii) if \( \Delta < 0 \), then \( f(x) \) has one real zero and two nonreal complex conjugate zeros.

Note that the discriminant \( \Delta(p) \) of \( f^{[p]}(x) \) is a continuous function of \( p \) and \( \Delta(1) > 0 \).
Assume that \( \Delta(s) < 0 \) for some \( s > 1 \). Then there exists \( 1 < r < s \) such that \( \Delta(r) = 0 \) by the continuity. By (ii), \( f^{[r]}(x) \) has a multiple zero and all its zeros are real. By the discussion in Case 2, \( f^{[r]}(x) = (f^{[r]})^{[s/r]}(x) \) has only real zeros, which contradicts (iii) since \( \Delta(s) < 0 \). Thus \( \Delta(p) \geq 0 \) for all \( p > 1 \), and so \( f^{[p]}(x) \) has only real zeros by (i) and (ii).

\[ \square \]

**Theorem 2.** If \( f(x) \in \mathbb{P}_n \) has only real zeros, then so does \( f^{[p]}(x) \) for \( p > P_n \), where

\[
P_n = \begin{cases} 
\frac{1}{\log_2(n+2)-\log_2 n}, & \text{if } n \text{ is even;} \\
\frac{1}{\log_2(n+3)-\log_2(n-1)}, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Proof.** Recall the Newton’s inequality: if \( f(x) = \sum_{i=0}^{n} a_i x^i \) has only real zeros, then

\[ a_i^2 \geq a_{i-1}a_{i+1} \frac{(i+1)(n-i+1)}{i(n-i)} \]

for all \( 0 < i < n \) (see, e.g., [5, p.104]). On the other hand, Kurtz [7, Theorem 1] showed that if \( g(x) = \sum_{i=0}^{n} b_i x^i \in \mathbb{P}_n \) and \( b_i^2 > 4b_{i-1}b_{i+1} \) for all \( 0 < i < n \), then all zeros of \( g(x) \) are real and distinct. So, to show that \( f^{[p]}(x) = \sum_{i=0}^{n} a_i^p x^i \) has only real zeros, it suffices to show that \( a_i^2 > 4a_{i-1}a_{i+1}^p \) for all \( 0 < i < n \).

Note that for \( 0 < i < n \), the function

\[ \frac{(i+1)(n-i+1)}{i(n-i)} = 1 + \frac{n+1}{i(n-i)} \]

attains the minimal value

\[ M_n := \begin{cases} 
\frac{(n+2)^2}{n+2}, & \text{if } n \text{ is even;} \\
\frac{n^2}{n-1}, & \text{if } n \text{ is odd.}
\end{cases} \]

Hence \( a_i^2 \geq M_n a_{i-1}a_{i+1} \), and so \( a_i^2 \geq M_p a_{i-1}a_{i+1}^p \) for all \( 0 < i < n \). Solve \( M_p > 4 \) to obtain

\[ p > P_n := \begin{cases} 
\frac{1}{\log_2(n+2)-\log_2 n}, & \text{if } n \text{ is even;} \\
\frac{1}{\log_2(n+3)-\log_2(n-1)}, & \text{if } n \text{ is odd.}
\end{cases} \]

Thus \( f^{[p]}(x) = \sum_{i=0}^{n} a_i^p x^i \) has only real zeros for \( p > P_n \). \[ \square \]

**Remark 1.** By (1), we have \( P_2 = 1; P_3 \approx 1.262; P_4 \approx 1.709; P_5 = 2; \ldots \)

A sequence \( \{a_i\}_{i=0}^{n} \) of positive numbers is called log-concave if \( a_{i-1}a_{i+1} \leq a_i^2 \) for \( 0 < i < n \). If the strict inequality holds for all \( i \), then we say that the sequence is strictly log-concave. Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{P}_n \). If \( f(x) \) has only real zeros, then \( \{a_i\}_{i=0}^{n} \) is strictly log-concave by the Newton’s inequality. By the same technique as used in the proof of Theorem 2, the assumption in Theorem 2 that \( f(x) \) has only real zeros can be weakened to requiring the strict log-concavity of the coefficients of \( f(x) \). We omit the details for the sake of brevity.

3
Theorem 3. Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{P}_n \). If \( \{a_i\}_{i=0}^{n} \) is strictly log-concave, then \( f[p](x) \) has only real zeros for \( p > \frac{2}{\ln 2} \beta_{\min} \), where \( \beta_{\min} = \min_{0<i<n} \frac{a_i^2}{a_{i-1}a_{i+1}} \).

3. Remarks

Let \( A \) be a finite or infinite matrix. We say that \( A \) is totally nonnegative (or TN, for short) if all minors of \( A \) are nonnegative. An infinite sequence \( a_0, a_1, a_2, \ldots \) is called a Pólya frequency (or PF) sequence if the Toeplitz matrix \( (a_{i-j})_{i,j \geq 0} \) is TN, where \( a_k = 0 \) for \( k < 0 \). We say that a finite sequence \( a_0, a_1, \ldots, a_n \) is PF if the infinite sequence \( a_0, a_1, \ldots, a_n, 0, 0, \ldots \) is PF. A classical result of Schoenberg and Edrei states that an infinite sequence \( a_0 = 1, a_1, a_2, \ldots \) is PF if and only if its generating function has the form

\[
\sum_{i \geq 0} a_i z^i = \prod_{j \geq 1} (1 + \alpha_j z) \prod_{j \geq 1} (1 - \beta_j z) e^{\gamma z},
\]

in some open disk centered at the origin, where \( \alpha_j, \beta_j, \gamma \geq 0 \) and \( \sum_{j \geq 1} (\alpha_j + \beta_j) < +\infty \) (see, e.g., Karlin [6, p. 412]). In particular, a finite sequence \( a_0, a_1, \ldots, a_n \) of positive numbers is PF if and only if the polynomial \( \sum_{i=0}^{n} a_i x^i \) has only real zeros [6, p. 399].

Let \( A = (a_{ij}) \) be a finite matrix with nonnegative elements. Define its \( p \)-th Hadamard power \( A[p] := (a_{ij}^p) \). Fallat and Johnson [1] showed, among other things, that if \( A \) is TN, then so is \( A[p] \) for sufficiently large \( p \), see also [2, Chapter 8]. This implies that if the polynomial \( f(x) \) has only real zeros, then so does its \( p \)-th Hadamard power for sufficiently large \( p \). However, our approach used in the proof of Theorem 2 is more direct and gives a lower bound for all polynomials of the same degree. Moreover, the known results in the literature are concerned with only finite matrices and can not be used in the proof of Theorem 1.

Acknowledgements

The authors thank the anonymous referee for his/her careful reading and valuable suggestions.

References