Periodic solutions to second order nonautonomous differential systems with gyroscopic forces

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Abstract
Existence of periodic solutions to second order differential systems with gyroscopic forces is considered via variational methods, where a generalized Ahmad–Lazer–Paul type condition is used. We do not impose the condition that the gyroscopic forces are small.

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1. Introduction
In this paper, we consider the $T$-periodic solution problem

\[
\begin{aligned}
\ddot{u} + A\dot{u} + \nabla F(t, u) &= 0, \quad \text{a.e. } t \in [0, T], \\
u(0) &= u(T), \quad \dot{u}(0) = \dot{u}(T),
\end{aligned}
\]

where $A$ is a real skew-symmetric $(N \times N)$ matrix, $F(t, u) : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following assumption:

(H) $F(t, u)$ is measurable in $t$ for each $u \in \mathbb{R}^N$, continuously differentiable in $u$ for a.e. $t \in [0, T]$, and there exist $a \in C(R^+, R^+)$ and $b \in L^1(0, T; R^+)$ such that $|\nabla F(t, u)| \leq a(|u|)b(t), \quad |\nabla^2 F(t, u)| \leq a(|u|)b(t)$, for all $u \in \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The term $A\dot{u}(t)$ means that the system has a gyroscopic force [1] and [2]. System (1) was mentioned by Ekeland [3] as possible extensions of his basic examples by modern variational methods, but without concrete work.

Under the restriction of the matrix being small in some sense, the homoclinic solution problem for the system was considered recently by variational methods by Buffoni and Sere [1], and Zhang and Yuan [4]. The periodic solution problem for the system was considered in Han [5,6] and recently in [7,8], where in [5,6] the following theorem (translated from Chinese) was established.

Theorem 0. Suppose that $F(t, u)$ satisfies the condition (H) and the following conditions:

(i) $A$ is an $(N \times N)$-skew-symmetric matrix and $\|A\| < 1$;
(ii) There exist $0 \leq \alpha < 1$, and $g(t), h(t) \in L^2(0, 2\pi; R^+)$ such that $|\nabla F(t, u)| \leq g(t)|u|^{\alpha} + h(t)$;
(iii) $\lim_{u \in \mathbb{R}^N, \|u\| \to \infty} \|u\|^{-2\alpha} \int_0^{2\pi} F(t, u)dt = +\infty(-\infty)$.

Then problem (1) has at least one $T=(2\pi)$-periodic solution.

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The first condition means that the matrix $A$ is small in some sense. When $A = 0$ (reduced to second order Hamiltonian system) and $\alpha = 0$ (bounded nonlinearity), the result was proved in [9]. For more results of second order Hamiltonian systems by variational methods, e.g. see [9,10]. The generalized Ahmad–Lazer–Paul type condition (iii) is used by several authors in the literature for dealing with other problems with a sublinear nonlinearity as in (ii), e.g. see [11] for some results and conditions directly imposed on the function $F(t, u)$.

All the results about the system mentioned above need the restriction of the matrix being small in some sense. That means that the gyroscopic force is only a small perturbation of the system. That is certainly a very special case for the effect of the force. In this paper, we present a result about the periodic solution problem (1) without this restriction.

2. Main result

As usually, we denote $H^1_T = \{ u(t) \mid u(t) \text{ is absolutely continuous on } [0, T], u(0) = u(T), \dot{u}(t) \in L^2(0, T; \mathbb{R}^n) \}$. The norms in $H^1_T$ and $C[0, T]$ are denoted by $\| \cdot \|$, $\| \cdot \|_c$ respectively, and $(\cdot)$ denotes the pairing between $H^{-1}_T$ and $H^1_T$ or the inner product in $H^1_T$. Define a functional on $H^1_T$,

$$\varphi(u) = \frac{1}{2} \int_0^T |\dot{u}|^2 dt - \frac{1}{2} \int_0^T (A \dot{u}, u) dt - \int_0^T F(t, u) dt.$$

Under the condition (H), the functional is $C^1$ in $H^1_T$ [7] with

$$\varphi'(u)v = \int_0^T \dot{u}v dt - \int_0^T (A \dot{u}, v) dt - \int_0^T (\nabla F(t, u), v) dt, \quad \forall u, v \in H^1_T$$

and the critical points correspond to the weak solutions of (1) [9].

Define $K : H^1_T \rightarrow H^1_T$ and $G : H^1_T \rightarrow H^1_T$ by $(Ku, v) = \int_0^T (A \dot{u}, v) dt$, $(Gu, v) = \int_0^T (u + \nabla F(t, u), v) dt$, $\forall u, v \in H^1_T$. It is well known that $G : H^1_T \rightarrow H^1_T$ is completely continuous [9]. The operator $K$ has the following properties.

**Proposition 1.** $K : H^1_T \rightarrow H^1_T$ is linear, self-adjoint and completely continuous.

**Proof.** For all $u, v \in H^1_T$, by the skew-symmetric property of $A$ and the definition of the weak derivatives [9], we have the self-adjoint property of $K$: $(Ku, v) = \int_0^T (A \dot{u}, v) dt = - \int_0^T (\dot{u}, Av) dt = \int_0^T (A \dot{u}, u) dt = (u, Kv)$.

In order to prove the complete continuity of $K$, we suppose that $u_n \rightharpoonup u (H^1_T)$. Hence $\dot{u}_n \rightharpoonup \dot{u} (L^2)$. Denote $\|\dot{u}_n\|_2 + \|\dot{u}\|_2 \leq M (M$ is a constant), $S = \{ v \in H^1_T \mid \|v\| = 1 \}$. $S$ is compact in $C[0, T]$ by the compact embedding $H^1_T \hookrightarrow \hookrightarrow C[0, T]$ and hence $S$ has a finite $\varepsilon$-net $\{ v_1, v_2, \ldots, v_k \}$ in $C[0, T]$ for all $\varepsilon > 0$. Hence there is $N > 0$, such that $\|\int_0^T (A \dot{u}_n - A u, v_j) dt\| < \varepsilon$ as $n \geq N$ for all $i \in \{1, 2, \ldots, k\}$. For every $v \in S$, choose $v_j$ such that $\|v - v_j\|_c \leq \varepsilon$. Hence, as $n \geq N$, we can get

$$\|Ku_n - Ku\| = \sup_{v \in S} \left| \int_0^T (A \dot{u}_n - A u, v) dt \right| \leq \sup_{v_j \in S} \left| \int_0^T (A \dot{u}_n - A u, v - v_j) dt \right| + \int_0^T (A \dot{u}_n - A u, v_j) dt \leq \|v - v_j\|_c \|A\| \sqrt{T} (\|\dot{u}_n\|_2 + \|\dot{u}\|_2) + \frac{\varepsilon}{2} \leq \|A\| \sqrt{T} M \varepsilon + \frac{\varepsilon}{2}.$$

Hence $K$ is completely continuous.

Suppose that the real skew-symmetric matrix $A$ has the following eigenvalues $\pm a_1, \ldots, \pm a_l, 0, \ldots, 0$, where $a_l > 0$. Set $\omega = \frac{2\pi}{T}$. We say that $A$ has the property (P) if one of the two properties holds (or equivalently (b) holds): (a) $a_l < 2$. $1 < j \leq s$; (b) If there is $j (1 < j \leq s)$ such that $a_j > 2$, then $\pm a_j \pm \sqrt{a_j^2 - 4} \neq 2k\omega$, $\forall k \in \mathbb{Z}$; if there is $j$ with $1 < j \leq s$ such that $a_j = 2$, we require that $T \neq 2n\pi$, $\forall n \in \mathbb{N}$. We remark that case (a) means that the matrix $A$ is small in some sense and it is a special case of case (b).

In order to state the main result clearly, we set $S(t, u) = F(t, u) + \frac{1}{2} u^2$. □

**Theorem 1.** Suppose that $F(t, u)$ satisfies the condition (H) and the matrix $A$ does not have the property (P). Moreover, assume

1. $(S)$ There exist $0 \leq \alpha < 1$ and $g(t), h(t) \in L^1(0, T; \mathbb{R}^+) \text{ such that } |\nabla S(t, u)| \leq g(t)|u|^\alpha + h(t);$
2. $(S_\varepsilon) \lim_{|u| \rightarrow \infty, u \in (K)} \|u\|^{-2\alpha} \int_0^T S(t, u) dt = +\infty (-\infty).$

Then the periodic problem (1) possesses at least one solution in $H^1_T$.

**Proof.** The matrix $A$ has the property (P) if and only if the linear $T$-periodic system (LP) $\ddot{u} + A \dot{u} - u = 0$, $u(0) = u(T)$, $\dot{u}(0) = \dot{u}(T)$ has only the trivial $T$-periodic solution.
Setting \( \dot{u} = v, Y = (u, v)^T \) and \( B = \begin{pmatrix} 0 & I_N \\ I_N & -A \end{pmatrix} \), we have the equivalent first order \( T \)-periodic system \( \dot{Y} = BY \).

\[
|\lambda I_N - B| = \begin{vmatrix} \lambda & -I_N \\ -I_N & \lambda I_N + A \end{vmatrix} = \begin{vmatrix} \lambda I_N & -0_N \\ -I_N & \lambda I_N + A - \frac{1}{\lambda} I_N \end{vmatrix} = \lambda^n \left( \lambda - \frac{1}{\lambda} \right) I_N + A = 0.
\]

It is obvious that \( B \) is invertible and hence the eigenvalues of \( B \) will be 1 or satisfy \( \lambda - \frac{1}{\lambda} = \pm \alpha j \), or \( \lambda = \pm \sqrt{-\alpha^2 + 4} \) \((1 \leq j \leq s)\). Now the claim follows from the standard results for linear \( T \)-periodic systems.

By the definition of \( K \), the functional \( \varphi \) can be written as \( \varphi(u) = \psi(u) - \int_0^T S(t, u)dt = \frac{1}{2}(I - K)u, u) - \int_0^T S(t, u)dt \).

By Proposition 1, we can decompose \( H^1 \) into \( H^1 = H^- \oplus H^0 \oplus H^+ \) such that there exists \( \delta > 0 \), \( \psi(u) \leq -\frac{1}{2} \|u\|^2 \) if \( u \in H^- \) and \( \psi(u) \geq \frac{1}{2} \|u\|^2 \) if \( u \in H^+ \), where \( H^0 = N(I - K) \) corresponds to the solution space of (LP) and \( H^+ \) \((H^-)\) corresponds to the subspace of \( H^1 \) spanned by the eigenspaces of \( K \) with eigenvalues smaller (bigger) than 1. Hence \( \dim H^+ = \dim H^- < \infty \).

Under the condition on the matrix \( A, H^0 \neq 0 \). If \( u \in H^0 \), we write \( u = u^+ + u^0 + u^- \), where \( u^+ \in H^+, u^0 \in \text{H}^0 \) and \( u^- \in H^- \).

We only prove the theorem where \( \dim H^- > 0 \) and \( (S_+) \) holds. The other cases can be similarly given or be easier.

**Claim.** \( \varphi \) satisfies P.S. condition.

Assume that \( \{u_k\} \subset H^1 \setminus \{\varphi(u_k)\} \) is bounded and \( \varphi'(u_k) \to 0 \) as \( k \to \infty \). We first prove that \( \{u_k\} \) is bounded in \( H^1 \). In the following, \( C \) denotes a \( (\alpha) \) universal constant and \( C(\varepsilon) \) is a constant dependent only of \( \varepsilon \).

By noticing condition (S) with \( 0 < \alpha < 1 \), embedding \( H^1 \hookrightarrow \hookrightarrow C[0, T] \) and Young's inequality, we can get for any \( \varepsilon > 0 \)

\[
\int_0^T (\nabla S(t, u_k), u_k^+)dt \leq \varepsilon \|u_k^+\|^2 + C(\varepsilon)\|u_k^0\|^{2\alpha} + C(\varepsilon)\|u_k^-\|^{2\alpha} + C(\varepsilon).
\]

Hence
\[
\varepsilon \|u_k^+\|^2 + C(\varepsilon) \geq C\|u_k^0\| \geq (\varphi'(u_k), u_k^+)
\]
\[
= ((I - K)u_k, u_k^+) - \int_0^T (\nabla S(t, u_k), u_k^+)dt
\]
\[
\geq \delta \|u_k^+\|^2 - \varepsilon \|u_k^0\|^2 - C(\varepsilon)\|u_k^0\|^{2\alpha} - C(\varepsilon)\|u_k^-\|^{2\alpha} - C(\varepsilon).
\]

Hence choosing \( \varepsilon > 0 \) small enough, we have \( \|u_k^+\|^2 \leq C\|u_k^0\|^{2\alpha} + C\|u_k^0\|^{2\alpha} + C \). Similarly, we can get \( \|u_k^-\|^2 \leq C\|u_k^0\|^{2\alpha} + C\|u_k^0\|^{2\alpha} + C \). Hence we can obtain \( \|u_k^+\|^2 \leq C\|u_k^0\|^{2\alpha} + C, \|u_k^-\|^2 \leq C\|u_k^0\|^{2\alpha} + C \); moreover, by the condition (S), we have

\[
\left| \int_0^T (S(t, u_k) - S(t, u_0^k))dt \right| \leq \left| \int_0^T \int_0^1 (\nabla S(t, u_0 + s(u_k^+ + u_0^-), u_k^+ + u_0^-))dsdt \right| \leq C\|u_0^k\|^{2\alpha} + C.
\]

Combining the above inequality and the estimates of \( \|u_k^+\|, \|u_k^-\| \) in term of \( \|u_0^k\| \), we can get
\[
-C \leq \varphi(u_k) = \frac{1}{2}((I - K)u_k, u_k) - \int_0^T (S(t, u_k) - S(t, u_0^k))dt - \int_0^T S(t, u_0^k)dt
\]
\[
\leq \|u_0^k\|^{2\alpha} \left[ C - \|u_0^k\|^{-2\alpha} \int_0^T S(t, u_0^k)dt \right] + C.
\]

Hence, by \( (S_+) \), \( (u_0^k) \) and moreover \( \{u_k\} \) is bounded in \( H^1 \). Now the P.S. condition follows from the standard argument.

It is known that \( \dim (H^0 \oplus H^-) < \infty \). For \( u \in H^0 \oplus H^- \), \( u = u^0 + u^- \), we have
\[
\varphi(u) = \frac{1}{2}((I - K)u^-, u^-) - \int_0^T (S(t, u) - S(t, u^0))dt - \int_0^T S(t, u^0)dt
\]
\[
\leq -\frac{\delta}{2} \|u^-\|^2 + \varepsilon \|u^-\|^2 + C(\varepsilon)\|u_0^0\|^{2\alpha} + C(\varepsilon) - \int_0^T S(t, u_0^0)dt \quad (\forall \varepsilon > 0),
\]

where we have used the condition (S). Choosing \( \varepsilon = \delta/4 \), by condition \((S_+)\), we get \( \varphi(u) \to -\infty \) as \( u \in H^0 \oplus H^- \), \( \|u\| \to \infty \).

On the other hand, for \( u = u^+ \in H^+ \), by the condition (S), we have
\[
\int_0^T S(t, u)dt = \int_0^T \int_0^1 (\nabla S(t, su), u)dsdt + \int_0^T S(t, 0)dt
\]
\[
\leq C\|u\|^{1+\alpha} + C\|u\| + C.
\]
Hence on $H^+$
\[ \varphi(u) \geq \frac{\delta}{2} \|u\|^2 - C \|u\|^{1+\alpha} - C \|u\| - C, \]
which implies that $\varphi(u)$ bounded below on $H^+$. By the saddle point theorem [9,12], the functional $\varphi(u)$ has at least one critical point in $H^1_T$.

**Remark 1.** When the matrix $A$ satisfies the property (P), then the condition $(S_{\pm})$ is empty and the condition (S) can be replaced by a weaker one
\[ \lim_{|u| \to \infty} \frac{\nabla S(t, u)}{|u|} = 0, \quad \text{uniformly for a.e. } t \in [0, T]. \]

In this case, the linear problem (LP) has only the trivial solution. The proof is similar but easier.

**Remark 2.** When the matrix $A$ relies on $t$, a similar framework can be constructed for the system. But it is unfortunate that the self-adjoint property of the operator $K$, some equalities about $K$ use it, is not preserved in general. But we can still establish some results as in [7] when $A(t)$ is small in some sense. When the matrix $A$ relies on $u$, some tools from exterior differential calculus will be used (skew-symmetric matrix is important in modern geometry) and the corresponding results will appear somewhere.

**References**